Moist Potential Vorticity and Up-Sliding Slantwise Vorticity Development *

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By using the moist potential vorticity equation derived from complete atmospheric equations including the effect of mass forcing, the theory of up-sliding slantwise vorticity development (USVD) is proposed based on the theory of slantwise vorticity development. When an air parcel slides up along a slantwise isentropic surface, its vertical component of relative vorticity is developed. Based on the theory of USVD, a complete vertical vorticity equation is expected with mass forcing, which explicitly includes the effect of both internal forcings and external forcings.

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The concept of potential vorticity (defined as $(\boldsymbol{\zeta}_a \cdot \nabla\theta)/\rho$, hereafter referred to as the Ertel PV), first introduced by Ertel,[1] is fundamental to our understanding of atmospheric dynamics. In a frictionless and adiabatic dry atmosphere, the Ertel PV is conserved. Besides its conservation, the Ertel PV has other two properties: invertibility in a balanced system and impermeability of PV substance (defined as $\boldsymbol{\zeta}_a \cdot \nabla\theta$). The Ertel PV is very useful in both diagnostic and prognostic studies of atmospheric phenomena.[2−5] Applications of the Ertel PV in the diagnosis of atmospheric motion have been summarized by Hoskins *et al.*[6]

In the studies of typhoons and torrential rains, the effects of moist air are discussed widely. The study of moist baroclinic atmosphere is one of the most important problems in summer of China.[7−9] When moisture is too important to be neglected, the definition and concept of PV are also available except that θ (potential temperature) is replaced by θ_e (equivalent potential temperature) in the expression of PV, and since the effect of moisture is involved, the new PV is called the moist potential vorticity (MPV). Bennetts and Hoskins[10] derived an equation for the variation of the so-called wet-bulb potential vorticity by using a set of equations with the Boussinesq approximation and concluded that the conditional symmetric instability was a possible cause of frontal rain belts. Based on the precise primitive equations, a similar variation equation for the MPV was also obtained by Wu and co-workers,[11−14] which shows that in a frictionless and adiabatic saturated atmosphere, MPV is conserved, and the theory of slantwise vorticity development (SVD) was proposed to study the development of the vertical component of vorticity in a moist baroclinic condition. According to the theory, vorticities are apt to develop near steep isentropic surfaces. In fact, since many kinds of weather systems in atmosphere do occur and develop near slantwise (moist) isentropic surfaces, it is applicable and necessary to investigate the evolution of the systems in the context of slantwise isentropic surfaces. A new form of vertical vorticity equation (CVVE) was also produced by Wu and Liu[14] from the definition of PV (MPV) and the PV (MPV) equation. Compared with the traditional vertical vorticity equation (TVVE),[15] the new equation has many advantages and is more applicable for diagnosis. To study torrential rains, an MPV equation was also derived by Gao *et al.*[16] with mass forcing and the impermeability of the MPV substance was proved. They concluded that the MPV substance anomaly induced by both heating and mass forcings during torrential rains will be a good tracer for tracking the region of torrential rains, which could be helpful in the forecast of torrential rains.

Some researchers[17,18] showed that up-sliding slantwise motions are always observed during the development and movement of oceanic frontal cyclones and rainstorms in Jiang-Huai valley of China, which actuates us to improve the theory of SVD to be suitable for this situation. In this Letter, an MPV equation is derived with diabatic heating, friction and mass forcing, and then the theory of up-sliding slantwise vorticity development (USVD) is proposed. Furthermore, a complete vertical vorticity equation (CVVE) is produced.

By taking vector product of momentum equation and noticing the fact that the vector product of a gradient is zero, we obtain the vorticity equation

$$\frac{\partial \boldsymbol{\zeta}_a}{\partial t} - \nabla \times (\boldsymbol{V} \times \boldsymbol{\zeta}_a) = \nabla p \times \nabla \alpha + \nabla \times \boldsymbol{F}, \qquad (1)$$

where $\boldsymbol{\zeta}_a = \nabla \times \boldsymbol{V} + 2\boldsymbol{\Omega}$ is the absolute vorticity, $\alpha = 1/\rho$, and ρ is the air density.

Including latent heating and other kinds of heating Q_d, thermodynamic equation can be written as

$$C_p \frac{T}{\theta} \frac{\mathrm{d}\theta}{\mathrm{d}t} = -L \frac{\mathrm{d}q}{\mathrm{d}t} + Q_d. \qquad (2)$$

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Upon applying natural logarithm and material derivative to both sides of the definition of equivalent potential temperature $\theta_e = \theta \mathrm{e}^{Lq/C_pT}$, putting it into the above thermodynamic Eq. (2) and omitting the high-order terms, we obtain

$$\frac{\mathrm{d}\theta_e}{\mathrm{d}t} = \frac{\theta_e}{C_pT}Q_d = Q. \tag{3}$$

By taking scalar product of Eq. (1) with $\nabla\theta_e$, we obtain

$$\begin{aligned}\alpha\frac{\mathrm{d}(\boldsymbol{\zeta}_a\cdot\nabla\theta_e)}{\mathrm{d}t} =& -\alpha(\boldsymbol{\zeta}_a\cdot\nabla\theta_c)\nabla\cdot\boldsymbol{V}\\ &+\alpha(\nabla p\times\nabla\alpha)\cdot\nabla\theta_e\\ &+\alpha\boldsymbol{\zeta}_a\cdot\nabla Q+\alpha\nabla\theta_e\cdot(\nabla\times\boldsymbol{F}).\end{aligned} \tag{4}$$

When it is raining, continuity equation can be expressed as[16]

$$\frac{\mathrm{d}\rho_a}{\mathrm{d}t} + \rho_a\nabla\cdot\boldsymbol{V} = -\nabla\cdot(\rho_r\boldsymbol{V}_t), \tag{5}$$

where $\boldsymbol{V}_t$ is the terminal velocity of precipitation particle; $\rho_a = \rho_d + \rho_v + \rho_c + \rho_r$ with ρ_a being the general density, and ρ_d, ρ_v, ρ_c, and ρ_r are the densities of dry air, vapour, cloud water, and rain water, respectively. There exist the following continuity equations

$$\frac{\mathrm{d}\rho_d}{\mathrm{d}t} + \rho_d\nabla\cdot\boldsymbol{V} = 0, \tag{6}$$

$$\frac{\mathrm{d}\rho_v}{\mathrm{d}t} + \rho_v\nabla\cdot\boldsymbol{V} = -Q_v \tag{7}$$

where Q_v is the transfer function from vapour to cloud water and rain water. Then air continuity equation is obtained by substituting Eq. (7) into Eq. (6), i.e.,

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \alpha\nabla\cdot\boldsymbol{V} + \alpha^2Q_v = \alpha\nabla\cdot\boldsymbol{V} + Q_n, \tag{8}$$

where $\alpha = 1/\rho$, $\rho = \rho_d + \rho_v$ and $Q_n = \alpha^2Q_v$ is contribution of mass forcing caused by precipitation and condensation to the variation of density.

By multiplying Eq. (8) with $\boldsymbol{\zeta}_a\cdot\nabla\theta_e$ on both the sides, substituting it into Eq. (4), defining $\boldsymbol{F}_\xi = \nabla\times\boldsymbol{F}$ as eddy friction diffusion and $P_m = \alpha\boldsymbol{\zeta}_a\cdot\nabla\theta_e$ as MPV, the MPV equation is found to be

$$\begin{aligned}\frac{\mathrm{d}(P_m)}{\mathrm{d}t} =& \alpha(\nabla p\times\nabla\alpha)\cdot\nabla\theta_e + \alpha\boldsymbol{\zeta}_a\cdot\nabla Q\\ &+\alpha\nabla\theta_e\cdot\boldsymbol{F}_\xi + (\boldsymbol{\zeta}_a\cdot\nabla\theta_e)Q_n,\end{aligned} \tag{9}$$

where $\alpha(\nabla p\times\nabla\alpha)\cdot\nabla\theta_e$ is the solenoid term. During torrential rains, θ_e is only the function of p and T for saturation, so the effect of solenoid term can be omitted. Here $\alpha\boldsymbol{\zeta}_a\cdot\nabla Q$ is the diabatic term without latent heating and the radiative heating is the main contributor since θ_e is used, $\alpha\nabla\theta_e\cdot\boldsymbol{F}_\xi$ and is the friction term. In free air, the effect of friction is not dominant. The final term $(\boldsymbol{\zeta}_a\cdot\nabla\theta_e)Q_n$, is the contribution of mass forcing caused by precipitation and condensation to the variation of MPV. By calculation,[19] the contribution of mass forcing should not be neglected during torrential rains (The variation of moist potential vorticity by mass forcing has an order of 10^{-5}PVU/s, which is identical to the variation by heat forcing. For simplicity, the figures are omitted).

Wu and Liu[11] advanced the theory of SVD. Since apparent slantwise up-sliding motions often exist just before severe storms, such as oceanic cyclones and rainstorms in the downwind direction, seen from both observations and numerical simulations,[17,18] a deduction of the theory of USVD is proposed and described in details in the following.

Z-coordinate is adopted here. USVD is described in Fig. 1, in which we assume that the parallel isentropic surfaces are horizontal or perpendicular outside the box $\mathrm{OFO_1E}$, but are bent as circles inside the box. For simplicity, we further assume that the gradient of isentropic surfaces, $\Delta\theta = \theta_n$, is constant. Also, circle b is defined, which is coaxial with the isentropic surfaces and keeps from $\theta+\Delta\theta$ at a constant distant $|\xi_n|$. From the definition of the Ertel PV, we obtain

$$P_E = \xi_n\theta_n = \xi_z\theta_z + \xi_s\theta_s, \tag{10}$$

then

$$\xi_z = \frac{\xi_n\theta_n - \xi_s\theta_s}{\theta_z} = \frac{P_E - \xi_s\theta_s}{\theta_z}, \quad (\theta_z \neq 0), \tag{11}$$

where $\xi_n = \alpha\zeta_n$ is the projection of $\boldsymbol{\xi}_a = \alpha\boldsymbol{\zeta}_a$ on $\boldsymbol{n}$ ($\boldsymbol{n}$ is the direction of $-\nabla\theta$), and $\xi_z = \alpha\zeta_z$, $\xi_s = \alpha\zeta_s$ is the vertical and horizontal component of $\boldsymbol{\xi}_a$, respectively; $\theta_n = |\nabla\theta|$, and θ_s and θ_z are the horizontal and vertical components of θ_n, respectively ζ_s; and ζ_z are the horizontal and vertical components of $\boldsymbol{\zeta}_a$.

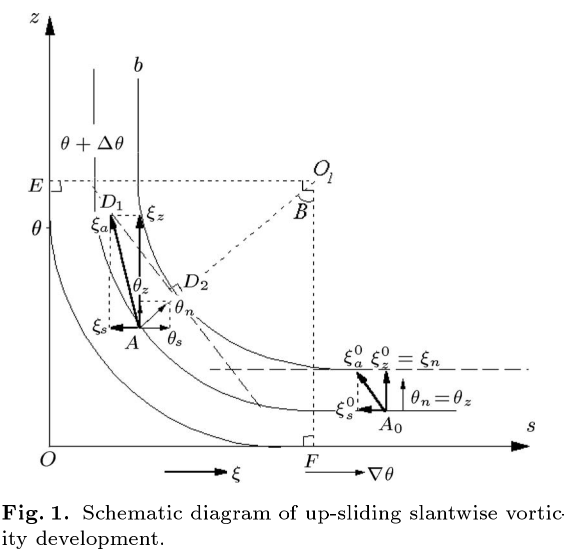

Fig. 1. Schematic diagram of up-sliding slantwise vorticity development.

For an air parcel A_0 moving on the isentropic surface $\theta+\Delta\theta$ leftward, when outside the box $\mathrm{OFO_1E}$, ζ_z

does not vary according to the BOX law[11] no matter what $\xi_s = \alpha\zeta_s = \alpha\partial V_m/\partial z$ is. When A_0 continues to move leftward on the isentropic surface $\theta + \Delta\theta$ into the box by sliding up an angle B (B is positive when one side deviates from $-z$ towards $-S$) to point A, the head point D_1 of $\boldsymbol{\xi}_a$ of the parcel A_0 should be located in a plane D_1D_2 which is a tangential place to circle b at point D_2 as shown in Fig. 1 according to the circumscribed plane law.[11] According to the figure, we have

$$\tan B = \frac{\theta_s}{\theta_z}, \quad \Big(-\frac{\pi}{2} < B < \frac{\pi}{2}\Big), \tag{12}$$

and $\boldsymbol{n}\cdot\boldsymbol{k} > 0$, then

$$\cos B = \frac{\theta_z}{\theta_n} > 0. \tag{13}$$

Introducing them into Eq. (11), we obtain

$$\xi_z = \frac{\xi_n}{\cos B} - \xi_s \tan B \quad \Big(|B| \neq \frac{\pi}{2}\Big). \tag{14}$$

In the northern hemisphere in the case of cyclogenesis, $\xi_n > 0$. If the following condition is satisfied, i.e.,

$$C_D = \frac{\xi_s\theta_s}{\theta_z} < 0, \tag{15}$$

Eq. (14) could be rewritten as

$$\xi_z = \frac{\xi_n}{\cos B} + |\xi_s \tan B| \quad \Big(|B| \neq \frac{\pi}{2}\Big), \tag{16}$$

where ξ_z increases with increasing $|B|$. When Eq. (15) is met (i.e., point A_0 is outside the box, $\theta_s = 0$, so originally C_D equals 0. Then Eq. (15) is equivalent to $\dot{C}_D < 0$), the vertical component of absolute vorticity of parcel A_0 should grow, and when the isentropic surfaces are sharply steep, ξ_z can become very large. Because the development of vorticity is due to the up-sliding of the air parcel along a slantwise isentropic surface, it can be referred as the USVD.

Upon applying material derivative to both the sides of Eq. (11), the CVVE for dry air can be obtained as

$$\frac{D_{\zeta_z}}{Dt} + \beta v + (f+\zeta_z)\nabla\cdot\boldsymbol{V} = \frac{1}{\alpha}\frac{D}{Dt}\Big[\frac{P_E}{\theta_z} - C_D\Big], \quad \theta_z \neq 0, \tag{17}$$

where P_E is the Ertel PV, $C_D = \xi_s\theta_s/\theta_z$ is the dry SVD index, θ_z is the vertical component of the gradient of potential temperature. If moist air is studied, the corresponding equation for moist air could be obtained as

$$\frac{D\zeta_z}{Dt} + \beta v + (f+\zeta_z)\nabla\cdot\boldsymbol{V} = \frac{1}{\alpha}\frac{D}{Dt}\Big[\frac{P_M}{\theta_{ez}} - C_M\Big] - \frac{Q_n(\zeta_z + f)}{\alpha}, \quad \theta_{ez} \neq 0, \tag{18}$$

where P_M represents the MPV, $C_M = \xi_s\theta_{\rm es}/\theta_{\rm ez}$ is the moist SVD index accordingly and $\theta_{\rm ez}$ is the vertical component of the gradient of equivalent potential temperature.

In summary, based on the theory of SVD, the theory of USVD has been proposed to study the variation of vorticity caused by slantwise up-sliding motions. From the definition of MPV and the MPV equation derived from complete atmospheric equations including the effect of mass forcing, the CVVE is expected-with mass forcing, which explicitly includes the effect of both internal forcings, such as variations of stability, baroclinicity and vertical shear of horizontal wind, and external forcings, such as diabatic heating, friction and mass forcing. When isentropic surfaces are flat, that is, $\theta_{\rm es}(\theta_s)$ equals to zero, it is easy to prove that the CVVE is identical to the TVVE. When isentropic surfaces are steep, the CVVE is more powerful. More details about the differences between these two equations will be reported elsewhere.

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